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The ray theory of ship waves and the class of streamlined ships

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A new theory is given for calculating the wave pattern and wave resistance of a ship moving at low Froude number F. It applies to ships of any width, either full-bodied or slender. In this theory, the waves travel along rays which start at source points, such as the bow and stern, on the water-line. They propagate with the speed of waves in deep water, but are also advected by the double body flow. This is the flow about the ship and its image in the undisturbed free surface. The phase of a wave at any point on a ray is the optical length of the ray from the source to that point. The amplitude is determined by an excitation coefficient, which determines its initial value, and by an integral along the ray. The total wave height at any point is the sum of the heights on all the rays through the point. The theory is incomplete because the excitation coefficients are known only for thin ships. As an illustration, the theory is applied to the thin ship case, and the results then agree with Michell's thin ship solution evaluated for F small.

A new class of ships, which we call streamlined ships, is introduced next. The usual linear free surface condition applies to the waves they produce. The ray theory is developed for these waves at low F, and it involves straight rays produced at all points on the rear half of the water-line. In addition, as an alternative to the ray theory, another method is presented for obtaining the waves at low F. It involves a Schrödinger-like equation in which distance along the ship's centre-line is the time-like co-ordinate.

1. Introduction

The 'ray theory of ship waves' is a new asymptotic theory, valid for low Froude number $F = U(gL)^{-\frac{1}{2}}$. Here U and L are the speed and length of the ship, and g is the acceleration due to gravity. The theory applies to ships of any width, full-bodied as well as slender. Thus it complements Michell's theory, which applies to thin ships at any Froude number. The ray theory was described in June 1974 at the Tenth Symposium on Naval Hydrodynamics (Keller 1974).

In the next section we shall describe the ray theory in physical terms. Then in §3 we shall present the equations from which the rays, phase, and amplitude can be calculated. In §4 we shall apply these equations to a thin ship and obtain results which agree with Michell's theory for that case. In §§5–10 we shall derive the equations of §3 formally from the equations of hydrodynamics.

This ray theory is patterned after the author's geometrical theory of diffraction (Keller 1953, 1962). That theory has been useful in electromagnetic theory, in acoustics, and in the study of surface gravity waves (Shen, Meyer & Keller 1968). It yields results which are very accurate for short waves, but the accuracy decreases as the

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wavelength λ increases until it is 5% or so for wavelengths as large as πL, where L is the dimension of an obstacle. This suggests that the present theory also may be useful for $\lambda \leqslant \pi L$, which corresponds to $2\pi F^2 \leqslant \pi$, or equivalently to $F \leqslant 2^{-\frac{1}{2}} \approx 0{\cdot}7$.

After developing the ray theory for ships of any shape, we shall introduce in §11 a particular class of ships which we call streamlined ships. These are ships for which a line parallel to the centre-line is nearly tangent to the hull at every point. This is the most inclusive class of ships for which it is valid to linearize the free surface boundary condition about that for a flat surface at any fixed Froude number, as in Michell's theory. However, the tangential flow condition holds on the actual hull, rather than on the midplane. The resulting linear boundary value problem has been studied by Brard (1972) and Noblesse (1978), but they did not consider the class of ships for which it is valid.

The ray theory and its derivation are both much simpler for streamlined ships than for general ships. The rays are straight lines and they are produced at all points on the rear half of the water-line, and not just at the special source points which play a role in the general case. By using these rays we can construct the wave pattern for any streamlined ship. One interesting result is that the wave pattern behind the ship on each side is determined by the hull shape on the *opposite* side! This is because the rays cross the centre-line behind the ship.

In addition to the ray theory for streamlined ships, we shall present another method for calculating the wave motion produced by such ships at low F. It involves a Schrödinger-like equation in which distance along the centre-line of the ship is the time-like co-ordinate. The advantage of this 'parabolic' equation is that the wave motion can be calculated by starting at the bow and marching toward the stern, finding the motion in the transverse plane at each step. The equation can also be approximated by a similar equation just on the free surface. These equations can be converted into Volterra integral equations on the hull and water-line, respectively, and the integral equations can also be solved by marching.

Finally we shall compare our results with those of other authors who have dealt with wavemaking at low Froude number. We shall see that Baba (1976) and Maruo (1977) unwittingly made approximations which restrict their results to streamlined ships at low Froude number. We shall also see that their results can be interpreted in terms of rays, as in our theory. However some of their rays travel *through the hull*! These rays, which are not present in our theory, are spurious. They show that some incorrect assumptions were made in deriving the integral representation from which the results were calculated. Noblesse (1978) also obtained this incorrect integral representation.

2. Physical description of the ray theory

The ray theory is based upon the fact that a ship moving at speed U produces waves of wavelength $\lambda = 2\pi U^2/g$. The ratio of this wavelength to the ship length L is $\lambda/L = 2\pi U^2/gL = 2\pi F^2$. Thus for small F the waves are short compared to L. Being short they must propagate along rays, and the rays must originate at the ship. Because short waves penetrate only a short distance beneath the surface, they must be produced at the water-line of the ship. But the waves produced by smooth portions of the water-line cancel one another by interference. Therefore, the rays are produced only at the bow, stern and other corners of the water-line, all of which we call sources. Thus the wave pattern consists of short waves which travel outward along rays from the sources.

To determine the rays, we note that aside from the wave motion, there is a non-uniform flow around the ship. When F is small, this flow is practically the 'double body' flow, which is the potential flow about the ship and its image in the undisturbed free surface. As the waves propagate, they are advected or carried by this flow. Thus their motion can be found by suitably combining the flow velocity with the propagation velocity of surface waves in water at rest.

The phase s of a wave at a point $\mathbf{x}$ is the optical length or travel time along a ray from its source to $\mathbf{x}$. The amplitude b involves an integral along the ray which expresses the energy balance in a narrow tube of rays. The 'initial value' of b is determined by an excitation coefficient E, which depends upon the shape of the hull near the source. The major unsolved problem of the present theory is the determination of E for non-thin ships.

In terms of the phase s and the amplitude b, the wave height on each ray is just the real part of $LF^3b\exp(iF^{-2}s)$. Then the total wave height at a point is the sum of these heights on all the rays through the point. Thus the wave height pattern exhibits interference in regions covered by two or more families of rays, and it is zero where there are no rays. The resistance can be found, as usual, by calculating the rate at which the waves carry energy to infinity.

We shall next present the equations of this theory.

3. Summary of the ray theory

Let $\mathbf{x} = (x, y)$, z, t be dimensionless co-ordinates with L the unit of length and $U^{-1}L$ the unit of time. The z axis is vertical while $\mathbf{x}$ is horizontal and the undisturbed water level is at $z = 0$. We denote the disturbed water surface by $z = F^2\eta(\mathbf{x}, t)$ and write η as the sum of two parts,

$$\eta(\mathbf{x},t) = \eta_0(\mathbf{x},t) + \sum_j b^j(\mathbf{x},t)\,F\exp\{iF^{-2}s^j(\mathbf{x},t) + \eta_0(\mathbf{x},t)\,[s^j_t + \nabla\phi\,.\,\nabla s^j]^2\} + O(F^2). \quad (3.1)$$

For a real function like η, the real part of the complex expression is always to be used. The first part η_0 is calculated from the double body potential $\phi(\mathbf{x}, z, t)$. As we shall show later, ϕ is the solution of (6.1)–(6.4), η_0 is given by (6.5), and η_0 does not represent waves. The second part is a sum of terms, one for each set of waves, or equivalently one for each ray through $\mathbf{x}$ at time t. In (3.1) $s^j(\mathbf{x},t)$ and $b^j(\mathbf{x},t)$ are the phase and amplitude on the jth such ray.

The rays are curves in the plane $z = 0$. Two families of them emanate from each source. One family is associated with the transverse wavefronts and the other with the longitudinal wavefronts (see figure 1). If a ray emanates from a source in a direction tangent to the water-line, it travels along the water-line. When it reaches another corner of the water-line, it produces diffracted rays. The diffracted rays also contribute to the wave pattern, but the amplitudes on them are of a smaller order in F than the amplitudes on the direct rays. Therefore, they do not contribute to the leading-order part of the wave pattern, so they can be omitted in calculating that part. Then typically just four rays, two from the bow and two from the stern, pass through each point in the wake.

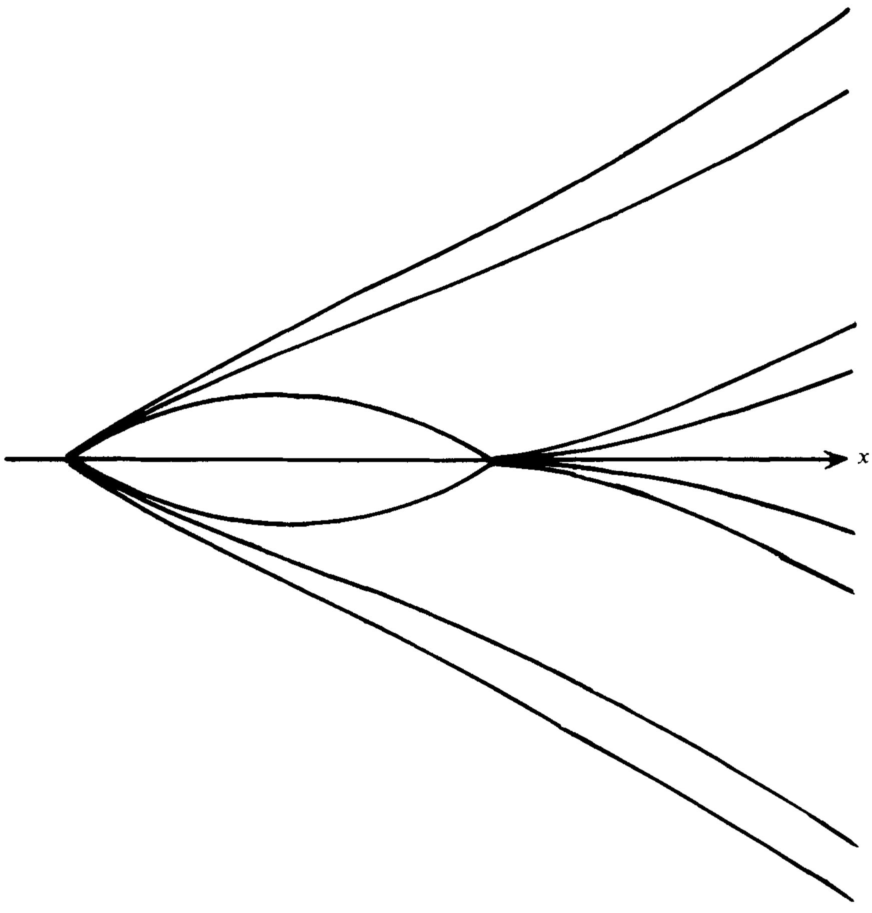

FIGURE 1. A sketch of some rays produced at the bow and stern of a ship moving to the left with constant velocity.

To define the rays, we begin with the equation for any phase function s, which is

$$(\nabla s)^2 = (s_t + \nabla\phi . \nabla s)^4. \tag{3.2}$$

This equation will be derived later as equation (8.2). In it $\phi = \phi(\mathbf{x}, 0, t)$ is the double body potential evaluated at $z = 0$. Equation (3.2) is just the dispersion equation for surface waves in deep water moving with the velocity $\nabla\phi$. To show this we introduce the wave vector $\mathbf{k} = \nabla s$ and the frequency $\omega = -s_t$. The dispersion equation states that the magnitude of $\mathbf{k}$ is equal to the square of the Doppler-shifted frequency $\omega - \mathbf{k} . \nabla\phi$. Squaring both sides of this relation yields $\mathbf{k}^2 = (\omega - \mathbf{k} . \nabla\phi)^4$, which is just (3.2). Since (3.2) involves only even powers of s, when s is a solution then so is $-s$.

The rays are the characteristic curves of (3.2). We shall write them in terms of a parameter σ as $\mathbf{x}(\sigma)$, $t(\sigma)$, and we must include $\mathbf{k}(\sigma)$, $\omega(\sigma)$ and $s(\sigma)$. We also introduce f, defined in (3.3), and write (3.2) in the form $f = 0$:

$$f(\mathbf{x}, t, \mathbf{k}, \omega) = 0; \quad f \equiv (\omega - \mathbf{k} . \nabla\phi)^4 - \mathbf{k}^2. \tag{3.3}$$

Then the rays are solutions of the following set of ordinary differential equations, in which a dot denotes $d/d\sigma$ (Courant & Hilbert 1962):

$$\dot{\mathbf{x}} = \nabla_k f, \quad \dot{\mathbf{k}} = \nabla_x f, \quad \dot{t} = -f_\omega, \quad \dot{\omega} = f_t, \quad \dot{s} = \mathbf{k}.\nabla_k f + \omega f_\omega. \tag{3.4}$$

The first six equations determine the rays and the last equation yields s.

In the special case of a ship moving with constant velocity, it is convenient to choose the origin of co-ordinates to be fixed in the ship with the x axis along the ship's path and with the ship velocity directed toward decreasing x. If the wave pattern is independent of t, we can omit the equations for t and ω from (3.4) and set $\omega = 0$. Then we can write (3.4) explicitly as

$$\dot{\mathbf{x}} = 4(\mathbf{k}.\nabla\phi)^3\nabla\phi - 2\mathbf{k}, \tag{3.5}$$

$$\dot{\mathbf{k}} = 4(\mathbf{k}.\nabla\phi)^3(\mathbf{k}.\nabla)\nabla\phi, \tag{3.6}$$

$$\dot{s} = 2k^2. \tag{3.7}$$

To solve these equations we need initial values. The initial value of $\mathbf{x}$ is the position of the source. The initial value of $\mathbf{k}$ can be found by writing $\mathbf{k} = (k_1, k_2)$ and choosing k_1 arbitrarily. Then (3.3) determines k_2. This equation is generally a quartic, so it generally has four solutions. But not all the solutions are admissible. Only those which are outgoing, i.e. which carry energy away from the source, can be used in constructing the wave pattern.

Wave energy propagates with the speed and direction given by the group velocity $c_g = \nabla_k \omega$. Therefore, only those rays for which c_g when evaluated near the source points away from it can be used. To evaluate c_g we solve (3.3) for ω and find

$$\omega = k^{\frac{1}{2}} + \mathbf{k}.\nabla\phi(\mathbf{x}, 0).$$

Differentiating yields

$$c_g = \tfrac{1}{2}k^{-\frac{3}{2}}\mathbf{k} + \nabla\phi(\mathbf{x}, 0). \tag{3.8}$$

Thus, only those rays are to be used for which this expression, evaluated near the source, points away from it.

The initial value of s can be chosen as $s = 0$ at the source. Then by integrating (3.7) along each outgoing ray we obtain

$$s(\sigma) = 2\int_0^\sigma \mathbf{k}^2(\sigma')\,d\sigma'. \tag{3.9}$$

The amplitude b at $\mathbf{x}(\sigma)$, $t(\sigma)$ is given by

$$b = -i(s_t + \nabla\phi.\nabla s)\,a(\sigma). \tag{3.10}$$

The function $a(\sigma)$, expressed by (10.6), involves E, the excitation coefficient of the source. The wave resistance R can be calculated by expressing the rate at which energy is carried away from the ship in terms of the preceding quantities, as in Keller & Ahluwalia (1976).

To apply the theory to the case of a constant velocity ship we must proceed as follows.

(a) Find the double body potential $\phi(\mathbf{x}, z)$ and evaluate it at $z = 0$.

(b) Evaluate $\eta_0(\mathbf{x}) = -\frac{1}{2}(\nabla\phi)^2$.

(c) Solve (3.5) and (3.6) for the outgoing rays $\mathbf{x}(\sigma)$, $\mathbf{k}(\sigma)$ from each source, using the above value of $\phi(\mathbf{x})$.

(*d*) Evaluate $s(\sigma)$ given by (3.9), using the solution for $\mathbf{k}(\sigma)$.

(*e*) Evaluate b given by (3.10). To do so a correction to ϕ, called ϕ_1, is needed in (10.6).

(*f*) Use the above results in (3.1) to get $\eta(\mathbf{x}, t)$, and in (54) of Keller & Ahluwalia (1976) to get R.

We must now consider whether there are rays emanating from smooth portions of the water-line, in addition to the rays from the source points considered above. If there were, they would yield a solution s of (3.2) with $s = 0$ on the water-line. For this solution s_t and the tangential component of ∇s are zero on the water-line, so (3.2) determines the normal component of ∇s there. Since $\nabla \phi$ is tangential to the water-line, (3.2) shows that $\nabla s = 0$. Thus the initial value of $\mathbf{k}$ is zero. In the case of a constant velocity ship it follows from (3.5) and (3.6) that there are no rays which leave the water-line with this initial value of $\mathbf{k}$. Thus no rays are produced at smooth portions of the water-line, as was stated in the preceding section.

4. Application of the ray theory to a thin ship

We shall now illustrate the use of the preceding theory by applying it to find the time independent wave pattern of a thin ship moving with constant velocity in the $-x$ direction. We assume that the bow intersects the undisturbed free surface at $x = -1, y = z = 0$ and the stern intersects it at $x = y = z = 0$ in a co-ordinate system fixed with respect to the ship. We shall first neglect the ship's beam, so that the ship lies in the x, z plane. Then it does not disturb the water at all, so the double body flow is just the unperturbed uniform flow with the potential $\phi(x, y, z) = x$. The corresponding free surface is the undisturbed surface, for which $\eta_0(x, y) = 0$. It can be verified easily that ϕ satisfies (6.1)–(6.4) and that η_0 is given by (6.5) with $\phi_t = 0$. This completes steps (*a*) and (*b*) of the procedure outlined above.

When we use $\phi(x, y, 0) = x$ in (3.6) it becomes $\dot{\mathbf{k}} = 0$. Thus $\mathbf{k}$ is constant on each ray. Next (3.5) becomes

$$\dot{x} = 4k_1^3 - 2k_1, \quad \dot{y} = -2k_2. \tag{4.1}$$

In view of the constancy of k_1 and k_2 along a ray, it follows from (4.1) that each ray is a straight line. To find k_2 we set $\omega = 0$ and $\phi = x$ in the dispersion equation (3.3), which becomes

$$k_1^2 + k_2^2 = k_1^4. \tag{4.2}$$

This has the two solutions

$$k_2 = \pm (k_1^4 - k_1^2)^{\frac{1}{2}}. \tag{4.3}$$

Then the solution of (4.1) for a ray starting at the stern $x = y = 0$ at $\sigma = 0$ is

$$x(\sigma) = (4k_1^3 - 2k_1)\,\sigma, \quad y(\sigma) = \mp 2(k_1^4 - k_1^2)^{\frac{1}{2}}\sigma. \tag{4.4}$$

The corresponding solution for a ray starting at the bow is also given by (4.4) with $x + 1$ in place of x.

To analyse the rays we introduce the polar co-ordinates r, α. Then from (4.3) we find that the slope angle of the ray with parameter k_1 is given by

$$\tan\alpha = \mp (k_1^2 - 1)^{\frac{1}{2}}/(2k_1^2 - 1). \tag{4.5}$$

From (4.5) we see that $\tan\alpha$ is a function of k_1^2 which is real for $k_1^2 \geqslant 1$. With the plus sign, it increases from $\tan\alpha = 0$ at $k_1^2 = 1$ to the maximum value $\tan\alpha^* = 2^{-\frac{3}{2}}$ at

$(k_1^*)^2 = \frac{3}{2}$, and then decreases to $\tan\alpha = 0$ at $k_1^2 = +\infty$. Thus, there are two values of k_1^2 corresponding to each value of $\tan\alpha$ in the range $0 < \tan\alpha < \tan\alpha^*$. This means that there are two rays from the stern in each direction within this range. By choosing the minus sign in (4.5), we find that there are also two rays in each direction in the range $-\tan\alpha^* < \tan\alpha < 0$.

The condition $-\tan\alpha^* < \tan\alpha < \tan\alpha^*$ is satisfied by α in either the range $-\alpha^* < \alpha < \alpha^*$ or the range $\pi-\alpha^* < \alpha < \pi+\alpha^*$. From (4.4) we see that when $k_1^2 > 1$ then $x(\sigma) > 0$ if k_1 and σ have the same sign while $x(\sigma) < 0$ if k_1 and σ have opposite signs. Thus, in the first case the ray will lie in the downstream sector $-\alpha^* < \alpha < a^*$, while in the second case it will lie in the other range. Only the rays in the downstream sector $-\alpha^* < \alpha < \alpha^*$ are outgoing, so we just include them and choose $k_1 > 0$ and $\sigma > 0$. This completes step (*c*).

Next, we use (3.9) for s, noting that $\mathbf{k}$ is constant along a ray, and (4.2) which gives $k^2 = k_1^4$. Thus

$$s(\sigma) = 2k_1^4\sigma. \tag{4.6}$$

Equations (4.4) and (4.6) represent $s(\mathbf{x})$ parametrically in terms of the parameters k_1 and σ. This completes step (*d*) of the programme. The two values of k_1 corresponding to each possible ray direction α yield two different values for s. They are exactly the phases of the transverse and longitudinal waves of the Kelvin ship wave pattern with its origin at the bow or the stern.

To compute the amplitude b given by (3.10), we note that $s_t = 0$ and $\nabla\phi\,.\,\nabla s = s_x = k_1$ in the present case. Thus $b = -ik_1 a(\sigma)$, with $a(\sigma)$ given by (9.9). To use (9.9) we must first find ϕ_1, the solution of (6.7)–(6.10). When we use $\phi = x$ and $\eta_0 = 0$ in (6.10), we find that the problem for ϕ_1 is homogeneous. Therefore its solution is $\phi_1 = 0$. Next (7.20) shows that $D = \partial_x$, so $Ds = s_x = k_1$, and then (9.4) yields $\gamma = 0$. As a consequence, the exponent in (9.9) becomes just the integral of $-\Delta s + 2(D^2 s)(Ds)^2$. In appendix B it is shown that

$$-\Delta s + 2(D^2 s)(Ds)^2 = -\frac{1}{2\sigma}. \tag{4.7}$$

Thus, the integral can be evaluated, $a(\sigma)$ can be found, and then b is given by

$$b = -ik_1 a(\sigma) = -ik_1 a(\sigma_0)(\sigma_0/\sigma)^{\frac{1}{2}}. \tag{4.8}$$

To find $a(\sigma_0)$ in (4.8) we use (10.5) with $f(\sigma_0) = \sigma_0^{\frac{1}{2}}$. In the present case $\nabla\phi = (1, 0)$, which is regular at the source, so we set $A = 1$ in (10.5). Therefore, $a(\sigma_0)\,\sigma_0^{\frac{1}{2}} = E(\alpha)$, where E is written as a function of the ray angle α. Then (4.8) becomes

$$b = -ik_1 E(\alpha)\,\sigma^{-\frac{1}{2}}. \tag{4.9}$$

This completes step (*e*), except that $E(\alpha)$ is still unknown.

Finally, we use the results (4.6) for s and (4.9) for b in (3.1) to get $\eta(\mathbf{x})$. Since $\eta_0 = 0$, (3.1) yields

$$F^2\eta(\mathbf{x}) = \sum_j -ik_1 F^3 E(\alpha)\,\sigma^{-\frac{1}{2}} \exp(2iF^{-2}k_1^4\sigma) + O(F^4). \tag{4.10}$$

There are at most four rays through $\mathbf{x}$, two from the bow and two from the stern, but there may be just two or none, depending upon the location of $\mathbf{x}$. The values of k_1 and σ depend upon the point $\mathbf{x}$, and whether the ray comes from bow or stern. These values are determined in terms of $\mathbf{x}$ by (4.4) for rays from the stern, and by (4.4) with $x+1$ in place of x for rays from the bow. There are also four coefficients

$E(\alpha)$, one for each of the rays. This completes the first part of step (f), the determination of η, with the $E(\alpha)$ still unknown.

Let us now compare our result (4.10) with the result obtained from the Michell theory for F small. That result, obtained by Keller & Ahluwalia (1976), is given by their equation (39) which is

$$h(\mathbf{x}) \sim \sum_{q,j=1}^{2} \frac{1}{r_q^{\frac{1}{2}}} E_{jq}(\alpha_q) \exp\{ikr_q\phi[\theta_j(\alpha_q), \alpha_q]\}, \quad |\alpha_2| < \alpha^*. \tag{4.11}$$

Here (r_q, α_q) are the polar co-ordinates of $\mathbf{x}$ with origin at the bow for $q = 1$ and at the stern for $q = 2$. The sum over j is over the two rays in the direction α_q, and the sum over q is over the two sources. We note that (4.11) holds for $|\alpha_2| < \alpha^*$, where four rays pass through each point.

In (4.11) $k = g/U^2 = L^{-1}F^{-2}$ and $E_{jq} = k^{-\frac{3}{2}}\tilde{E}_{jq} = L^{\frac{3}{2}}F^3\tilde{E}_{jq}$, where $\tilde{E}_{jq}$ is dimensionless. Furthermore, it follows from the definitions of k_1, σ and of ϕ in (4.11) that for either bow or stern

$$\frac{\sigma}{L^{-1}r_q} = \frac{\phi[\theta_j(\alpha_q), \alpha_q]}{2k_1^4}, \quad k_1 = \sec\theta_j(\alpha_q). \tag{4.12}$$

When these expressions are used in (4.11), it becomes

$$L^{-1}h(\mathbf{x}) \sim \sum_{j,q=1}^{2} \frac{\phi^{\frac{1}{2}}[\theta_j(\alpha_q), \alpha_q]\,\tilde{E}_{jq}(\alpha_q)}{2^{\frac{1}{2}}k_1^2} \frac{F^3}{\sigma^{\frac{1}{2}}} \exp(iF^{-2}2k_1^4\sigma). \tag{4.13}$$

Comparison shows that (4.10) and (4.13) depend in exactly the same way on F and σ. They become identical if we define $E(\alpha)$ by

$$E(\alpha) = -2^{-\frac{1}{2}}\cos^2\theta_j(\alpha)\,\phi^{\frac{1}{2}}[\theta_j(\alpha), \alpha]\,\tilde{E}_{jq}(\alpha). \tag{4.14}$$

All the functions in (4.14) are defined in Keller & Ahluwalia (1976), equations (22), (23) and (40).

We have now found exact agreement between the wave height given by the present theory, when specialized to a thin ship, and that given by the Michell theory when specialized to small F. This agreement implies that R given by the present theory also agrees exactly with the result of the Michell theory for small F, so we shall not calculate R. The comparison has also enabled us to determine E for a thin ship. It is given by (4.14).

We shall now show how the preceding theory can be derived from the hydrodynamic boundary value problem which is formulated in §5.

5. Exact formulation of the problem of flow past a ship

We wish to find the velocity potential $\Phi'(\mathbf{x}', z', t')$ and the free upper surface $z' = \eta'(\mathbf{x}', t')$ of an inviscid incompressible fluid, due to the motion of a ship. Here $\mathbf{x}' = (x', y')$ and z' are horizontal and vertical co-ordinates, respectively, and t' is time. Let the lower boundary of the fluid be the rigid surface $z' = -h'(\mathbf{x}')$ and let $B(t')$ denote the surface of the ship at time t'. The normal component of velocity of the ship at the point $\mathbf{x}', z', t'$ on $B(t')$ is denoted by $v'(\mathbf{x}', z', t')$. We now introduce the corresponding dimensionless quantities Φ, η, etc. defined by

$$\Phi' = UL\Phi, \quad \eta' = F^2L\eta, \quad (\mathbf{x}', z', h') = L(\mathbf{x}, z, h), \quad t' = U^{-1}Lt, \quad v' = Uv. \tag{5.1}$$

In terms of these dimensionless quantities, the relevant equations of hydrodynamics and boundary conditions are

$$\Delta\Phi = 0, \qquad -h(\mathbf{x}) \leqslant z \leqslant F^2\eta(\mathbf{x},t), \tag{5.2}$$

$$\partial\Phi/\partial n = v \qquad \text{on} \quad B(t), \tag{5.3}$$

$$\partial\Phi/\partial n = 0, \qquad z = -h(\mathbf{x}), \tag{5.4}$$

$$\Phi_z = F^2(\eta_t + \nabla\Phi\,.\,\nabla\eta), \quad z = F^2\eta(\mathbf{x},t), \tag{5.5}$$

$$\eta = -\Phi_t - \tfrac{1}{2}(\nabla\Phi)^2, \qquad z = F^2\eta(\mathbf{x},t). \tag{5.6}$$

In addition Φ and η must satisfy suitable conditions at infinity and at some initial time.

6. The double body potential and the F^2 correction

The solution Φ, η of (5.2)–(5.6) depends upon F. To find this solution for F small, we first set $F = 0$ in (5.2)–(5.6) and write ϕ, η_0 for the solution at $F = 0$. In this way we get

$$\Delta\phi = 0, \qquad -h(\mathbf{x}) \leqslant z \leqslant 0, \tag{6.1}$$

$$\partial\phi/\partial n = v \qquad \text{on} \quad B(t), \tag{6.2}$$

$$\partial\phi/\partial n = 0, \qquad z = -h(\mathbf{x}), \tag{6.3}$$

$$\phi_z = 0, \qquad z = 0, \tag{6.4}$$

$$\eta_0 = -\phi_t - \tfrac{1}{2}(\nabla\phi)^2, \quad z = 0. \tag{6.5}$$

The solution ϕ of (6.1)–(6.4) is called the double body potential because it can be found by reflecting $B(t)$ and the flow region in the plane $z = 0$. In this extended region, the flow is that past $B(t)$ and its image, which constitute the 'double body'. Once ϕ is found η_0 is given by (6.5).

For F^2 small but not zero, we write

$$\Phi = \phi + F^2\phi_1 + \dots, \qquad \eta = \eta_0 + F^2\eta_1 + \dots\,. \tag{6.6}$$

We substitute (6.6) into (5.2)–(5.6) and equate to zero the coefficients of F^2 to obtain

$$\Delta\phi_1 = 0, \qquad -h(\mathbf{x}) \leqslant z \leqslant 0, \tag{6.7}$$

$$\partial\phi_1/\partial n = 0 \qquad \text{on} \quad B(t), \tag{6.8}$$

$$\partial\phi_1/\partial n = 0, \qquad z = -h(\mathbf{x}), \tag{6.9}$$

$$\phi_{1z} = \eta_{0t} + \nabla\phi\,.\,\nabla\eta_0 - \phi_{zz}\eta_0, \quad z = 0, \tag{6.10}$$

$$\eta_1 = -\phi_{1t} - \nabla\phi\,.\,\nabla\phi_1, \qquad z = 0. \tag{6.11}$$

Once (6.7)–(6.10) are solved for ϕ_1, then (6.11) yields η_1.

Neither ϕ, η_0 nor ϕ_1, η_1 exhibits wave motion, and the continuation of the power series expansions (6.6) will not do so either. This is related to the fact that the solution is not analytic in F at $F = 0$, but has an essential singularity there. Therefore Φ and η cannot be power series, but must contain non-analytic terms. In appendix A some simple examples are given to indicate how such terms arise.

7. Wave motion

To describe the wave motion, we shall modify (6.6) and write

$$\Phi = \phi + F^2\phi_1 + F^3 \exp[iF^{-2}s](a + F^2 a_1) + \ldots, \tag{7.1}$$

$$\eta = \eta_0 + F^2\eta_1 + F\exp[iF^{-2}s(\mathbf{x}, F^2\eta, t)](b + F^2 b_1) + \ldots\ . \tag{7.2}$$

The complex functions s, a, a_1, b and b_1 are to be found, while the real functions ϕ, η_0, ϕ_1 and η_1 satisfy the equations (6.1)–(6.11), and will be considered to be known. It is understood that Φ and η are given by the real parts of the right sides of (7.1) and (7.2). These equations do not hold in a boundary layer near the ship.

In general Φ and η contain a sum of wave terms with different phases and amplitudes, corresponding to the different waves passing through $\mathbf{x}$ at time t. But these waves satisfy linear equations, to the order we shall determine them, as is shown below. Therefore it suffices to write just one wave term in order to derive the equations for it. Of course in calculating Φ and η all the waves must be included.

Some reasons for assuming the particular form given above for the wave motion are the following. First of all, the dimensionless wavenumber k associated with the Froude number F is $k = F^{-2}$. Since we are considering F to be small, k is large. Now linear wave motions with k large are always proportional to e^{iks}, which accounts for this factor in (7.1). The corresponding factor in (7.2) is evaluated on the free surface $z = F^2\eta$, as is to be expected from (5.6). The factor F multiplying the exponential in (7.2) results from the fact that, as we shall see, b_0 behaves like $r^{-\frac{1}{2}}$ near the source, where r is distance from the source. This singular behaviour should be independent of L, so dimensional analysis dictates that η must behave like $(kr)^{-\frac{1}{2}}$. Since $k^{-\frac{1}{2}} = F$, this explains the factor F in (7.2). The factor F^3 in (7.1) is needed so that in (5.6) derivatives of the wave term in (7.1) can balance the term proportional to F in η. However, the subsequent analysis also holds if the wave terms in (7.1) and (7.2) have the factors $F^{3+\xi}$ and $F^{1+\xi}$ respectively, for any positive number ξ.

We must now substitute (7.1) and (7.2) into (5.2)–(5.6) and equate to zero the coefficients of the wave factor $\exp(iF^{-2}s)$. Because this calculation is a bit complicated, we shall break it up into the following three steps, which are equivalent to direct substitution.

(1) We write $\Phi = \overline{\phi} + \phi' + \ldots$, $\eta = \overline{\eta} + \eta' + \ldots$, and linearize the equations with respect to ϕ' and η'.

(2) We substitute $\phi' = F^3\exp(iF^{-2}s)A$, $\eta' = F\exp[iF^{-2}s(\mathbf{x}, F^2\overline{\eta}, t)]B$ into the linear equations.

(3) We set $\overline{\phi} = \phi + F^2\phi_1 + \ldots$, $\overline{\eta} = \eta_0 + F^2\eta_1 + \ldots$, $A = a + F^2 a_1 + \ldots$, $B = b + F^2 b_1 + \ldots$ in the resulting equations and equate coefficients of each power of F in each equation.

The first step yields

$$\Delta\phi' = 0, \quad -h \leqslant z \leqslant F^2\overline{\eta}, \tag{7.3}$$

$$\partial\phi'/\partial n = 0 \quad \text{on} \quad B(t), \tag{7.4}$$

$$\partial\phi'/\partial n = 0, \quad z = -h, \tag{7.5}$$

$$\phi'_z + F^2\eta'\overline{\phi}_{zz} = F^2(\eta'_t + \nabla\overline{\phi}\,.\,\nabla\eta' + \nabla\phi'\,.\,\nabla\overline{\eta}) + F^4\eta'\nabla\overline{\eta}\,.\,\nabla\overline{\phi}_z, \quad z = F^2\overline{\eta}, \tag{7.6}$$

$$\eta' = -\phi'_t - F^2\overline{\phi}_{tz}\eta' - \nabla\overline{\phi}\,.\,\nabla\phi' - F^2\eta'\nabla\overline{\phi}\,.\,\nabla\overline{\phi}_z, \quad z = F^2\overline{\eta}. \tag{7.7}$$

As the second step, we substitute

$$\phi' = F^3 \exp(iF^{-2}s)\,A$$

and

$$\eta' = F \exp[iF^{-2}s(\mathbf{x}, F^2\eta, t)]\,B$$

into (7.3), (7.4), (7.6) and (7.7) to get

$$-(\nabla s)^2 A + iF^2(2\nabla s\,.\,\nabla A + A\Delta s) + F^4\Delta A = 0, \quad -h \leqslant z \leqslant F^2\bar{\eta}, \tag{7.8}$$

$$i(\partial s/\partial n)\,A + F^2(\partial A/\partial n) = 0, \tag{7.9}$$

$$\begin{aligned}-is_z A + i(s_t + \bar{\phi}_x s_x + \bar{\phi}_z s_z)\,B + F^2[\bar{D}B + is_z B\bar{D}\bar{\eta} + iA\nabla\bar{\eta}\,.\,\nabla s - A_z - B\bar{\phi}_{zz}]\\ + F^4[B\nabla\bar{\eta}\,.\,\nabla\bar{\phi}_z + \nabla\bar{\eta}\,.\,\nabla A] = 0, \quad z = F^2\bar{\eta},\end{aligned} \tag{7.10}$$

$$B + i(\bar{D}s)\,A + F^2[B\bar{D}\bar{\phi}_z + \bar{D}A] = 0, \quad z = F^2\bar{\eta}. \tag{7.11}$$

Here we have introduced $\bar{D}$, defined by

$$\bar{D} = \partial_t + \nabla\bar{\phi}\,.\,\nabla. \tag{7.12}$$

We shall consider (7.5) later.

Now as the third step, we substitute $A = a + F^2a_1 + \ldots$, etc. into (7.8)–(7.11) and equate to zero the coefficients of the two lowest powers of F^2. However in (7.9) we need only the coefficient of the lowest power of F^2. In treating (7.10) and (7.11) we must also expand the functions of z, which are evaluated at $z = F^2\bar{\eta}$, about $z = 0$. This step yields the following equations:

$$(\nabla s)^2 = 0, \qquad -h \leqslant z \leqslant 0; \tag{7.13}$$

$$2\nabla s\,.\,\nabla a + a\Delta s = 0, \quad -h \leqslant z \leqslant 0; \tag{7.14}$$

$$(\partial s/\partial n)\,a = 0 \qquad \text{on} \quad B(t); \tag{7.15}$$

$$-s_z a + (Ds)\,b = 0, \quad z = 0; \tag{7.16}$$

$$\begin{aligned}-s_z a_1 + (Ds)\,b_1 = -(\phi_{1x}s_x + \phi_{1z}s_z)\,b + i[Db + is_z bD\eta_0 + ia\nabla\eta_0\,.\,\nabla s - a_z - b\phi_{zz}]\\ + \eta_0\,\partial_z[s_z a - (s_t + \phi_x s_x + \phi_y s_y)\,b], \quad z = 0;\end{aligned} \tag{7.17}$$

$$b + i(Ds)\,a = 0, \quad z = 0; \tag{7.18}$$

$$b_1 + i(Ds)\,a_1 = -i(\nabla\phi_1\,.\,\nabla s)\,a - bD\phi_z - Da - i\eta_0\,\partial_z(aDs), \quad z = 0. \tag{7.19}$$

Here

$$D = \partial_t + \nabla\phi\,.\,\nabla. \tag{7.20}$$

On $z = 0$, (6.4) shows that $\phi_z = 0$, and then $D = \partial_t + \phi_x\partial_x + \phi_y\partial_y$. We have used this fact above, except where D is differentiated. In the following sections we shall deduce the consequences of the equations (7.13)–(7.19).

8. The dispersion equation

Equations (7.16) and (7.18) are two homogeneous linear equations for a and b, which have a non-trivial solution only if the determinant of the coefficient matrix vanishes. The vanishing of this determinant yields

$$s_z = -i(s_t + \nabla\phi\,.\,\nabla s)^2, \quad z = 0. \tag{8.1}$$

Now we use (8.1) to eliminate s_z from (7.13) on $z = 0$. In this way we obtain

$$(s_x^2 + s_y^2)^{\frac{1}{2}} = (s_t + \phi_x s_x + \phi_y s_y)^2, \quad z = 0. \tag{8.2}$$

This is a first-order partial differential equation for $s(\mathbf{x}, 0, t)$, which we call the 'dispersion equation'.

This dispersion equation is that for an infinitely deep fluid because it was derived for short waves. Only the velocity $\nabla\phi$ at $z = 0$ enters it because the wave motion decays exponentially with depth. This is evident from (8.1), which shows that s_z is negative imaginary at $z = 0$. Then the factor $\exp(iF^{-2}s)$ decays rapidly with increasing distance below the surface. Another consequence of this exponential decay is that $\partial\phi'/\partial n$ is exponentially small at $z = -h$, so (7.5) is satisfied asymptotically. That is why we omitted it above.

In § 1 we showed how to solve (8.2) for the phase function s on $z = 0$, by the method of characteristics. The solution can be extended to $z < 0$ by using its values on $z = 0$ as initial conditions for (7.13). That equation can also be solved by rays, which are complex straight lines, but we shall not need the solution. However, we do need s at $z = F^2\eta$, since that is where it is evaluated in (7.2). To find it we shall use its Taylor expansion to two terms:

$$s(\mathbf{x}, F^2\eta, t) = s(\mathbf{x}, 0, t) + F^2\eta(\mathbf{x}, t)\, s_z(\mathbf{x}, 0, t) + O(F^4). \tag{8.3}$$

Now s_z is given by (8.1) and η is given by (7.2). Therefore (8.3) becomes

$$s(\mathbf{x}, F^2\eta, t) = s(\mathbf{x}, 0, t) - iF^2\eta_0(\mathbf{x}, t)\,[s_t + \nabla\phi\,.\,\nabla s]^2 + O(F^4). \tag{8.4}$$

We now use (8.4) in (7.2) and retain only the leading terms to obtain

$$\eta(\mathbf{x}, t) = \eta_0(\mathbf{x}, t) + Fb(\mathbf{x}, t)\exp\{iF^{-2}s(\mathbf{x}, 0, t) + \eta_0(\mathbf{x}, t)\,[s_t + \nabla\phi\,.\,\nabla s]^2\} + \dots\,. \tag{8.5}$$

If we wish to construct the waves on surface diffracted rays, we must consider the boundary condition for s on $B(t)$. Then we find, from (7.15), if $a \neq 0$ on $B(t)$, that

$$\partial s/\partial n = 0 \quad \text{on} \quad B(t). \tag{8.6}$$

We can rewrite this as $n\,.\,\nabla s = 0$ and then on $z = 0$ we can use (8.1) to eliminate s_z. In this way we get

$$n_x s_x + n_y s_y - in_z(s_t + \phi_x s_x + \phi_y s_y)^2 = 0 \quad \text{on} \quad W(t). \tag{8.7}$$

Here (n_x, n_y, n_z) are the components of the normal to $B(t)$, and $W(t)$ is the water-line, which is the curve of intersection of $B(t)$ and the plane $z = 0$. In (8.7) only the values of s on the surface $z = 0$ are involved.

9. The transport equation

From (7.18) we have

$$b = -i(s_t + \nabla\phi\,.\,\nabla s)\,a, \quad z = 0. \tag{9.1}$$

This determines b in terms of a on $z = 0$. To find a on $z = 0$, we observe that (7.17) and (7.19) are a pair of inhomogeneous linear equations for a_1 and b_1 with the same coefficient matrix as that in (7.16) and (7.18). Since that matrix is singular, the right sides of (7.17) and (7.19) must satisfy a solvability condition in order that these equations have a solution. This condition, obtained by subtracting Ds times (7.19) from (7.17) and simplifying, is

$$\begin{aligned}(2\nabla\phi_1.\nabla s-\phi_{1z}s_z)\,b-iDb+s_z bD\eta_0 \\ +a\nabla s.\nabla\eta_0+ia_z+ib\phi_{zz}-b(Ds)\,(D\phi_z) \\ -(Ds)\,(Da)+\eta_0 b(2Ds+\nabla s.\nabla\phi_z)-s_{zz}\eta_0 a=0, \quad z=0.\end{aligned} \tag{9.2}$$

This equation is of the form

$$-iDb-(Ds)\,(Da)+ia_z+\gamma b+(\nabla s.\nabla\eta_0-\eta_0 s_{zz})\,a=0, \tag{9.3}$$

where γ is the coefficient of b in (9.2). Since $\phi_z=0$ on $z=0$, as (6.4) shows, γ can be simplified to the form

$$\gamma=2\nabla\phi_1.\nabla s-\phi_{1z}s_z+s_z D\eta_0+i\phi_{zz}+2\eta_0 Ds+\eta_0 s_z\phi_{zz}, \quad z=0. \tag{9.4}$$

Furthermore b can be eliminated from (9.3) by means of (9.1), and a_z can be eliminated by using (7.14), which yields

$$a_z=-\frac{1}{2s_z}\,(\nabla_2 s.\nabla_2 a+a\Delta s). \tag{9.5}$$

Here $\nabla_2=(\partial_x,\partial_y)$. Then since $s_z=-i(Ds)^2$ according to (8.1), we can write (9.3) as follows:

$$-2(Ds)\,(Da)+(Ds)^{-2}\nabla_2 s.\nabla_2 a_0+[\tfrac{1}{2}(Ds)^{-2}\Delta s-D^2s-i\gamma Ds+\nabla s.\nabla\eta_0-\eta_0 s_{zz}]\,a=0, \quad z=0. \tag{9.6}$$

We can write (9.6) as an ordinary differential equation for $a_0(\sigma)$ by introducing the directional derivative

$$d/d\sigma=-4(Ds)^3 D+2\nabla_2 s.\nabla_2, \quad z=0. \tag{9.7}$$

Then (9.6) becomes

$$da/d\sigma+2(Ds)^2[\tfrac{1}{2}(Ds)^{-2}\Delta s-D^2s-i\gamma Ds+\nabla s.\nabla\eta_0-\eta_0 s_{zz}]\,a=0, \quad z=0. \tag{9.8}$$

This equation is called the transport equation for a. From equation (3.4) it follows that the derivative (9.7) is along a characteristic curve of the dispersion equation (8.2). Since we call such a curve a 'ray', (9.8) is an equation for a along a ray. Its solution is

$$\begin{aligned}a(\sigma)=a(\sigma_0)\exp\left\{-2\int_{\sigma_0}^{\sigma}(Ds)^2[\tfrac{1}{2}(Ds)^{-2}\Delta s-D^2s-i\gamma Ds\right. \\ \left.+\nabla s.\nabla\eta_0-\eta_0 s_{zz}]\,d\sigma'\right\}, \quad z=0.\end{aligned} \tag{9.9}$$

The result (9.9) for $a(\sigma)$ along a ray can be used in (9.1) to determine the wave amplitude b. This completes the determination of η, which is given by (8.5). However the initial value $a(\sigma_0)$ in (9.9) has not yet been determined, so we shall now consider how to find it.

10. Excitation coefficients

The initial value $a(\sigma_0)$ must be determined by considering wave production at the source $\sigma_0=0$. The source is a focal point of rays and a stagnation point of the flow, so we must expect $a(\sigma_0)$ to be singular there. For thin ships (4.9) shows that $a(\sigma)$ is proportional to $\sigma^{-\frac{1}{2}}$, and the corresponding result for more general ships can be obtained from (9.9). Let us assume that $a(\sigma)$ becomes infinite like $1/f(\sigma)$ as σ tends to zero, where $f(\sigma)$ is a function which vanishes at $\sigma = 0$. The function $f(\sigma)$ can be calculated from (9.9), but we shall not calculate it here.

We now rewrite (9.9) with the factor $f(\sigma_0)[f(\sigma_0)]^{-1}$ on the right side, and then we take the limit of both sides as σ_0 tends to zero. The left side $a(\sigma)$ is independent of σ_0, while the right side can be written as a product of two limits as follows:

$$a(\sigma) = [\lim_{\sigma_0\to 0} f(\sigma_0)\, a(\sigma_0)] \left[\lim_{\sigma_0\to 0} \frac{1}{f(\sigma_0)} \exp\left\{-2\int_{\sigma_0}^{\sigma} (Ds)^2 \times [\tfrac{1}{2}(Ds)^{-2}\Delta s - D^2 s - i\gamma Ds + \nabla s\,.\,\nabla\eta_0 - \eta_0 s_{zz}]\, d\sigma'\right\}\right]. \quad (10.1)$$

Since we have assumed that $a(\sigma_0)$ becomes infinite like $1/f(\sigma_0)$, the first limit in (10.1) exists. Then because the left side is independent of σ_0, the second limit also exists. In fact $f(\sigma)$ can be characterized by the requirement that this second limit exist.

In order to use (10.1), we must determine the first limit in it. Since $\sigma_0 = 0$ denotes the source point, where the waves are produced, this limit must be determined by the potential ϕ and the shape of the hull near the source, and by the angle α at which the ray leaves the source. Near the source the hull has a tangent cone with its apex at the source. This tangent cone and the plane $z = 0$ bound a conical domain Γ within which ϕ is defined near the source. To find the behaviour of ϕ there we write

$$\phi(r, \boldsymbol{\omega}) = A r^{\beta} \psi(\boldsymbol{\omega}) + \dots\ . \quad (10.2)$$

Here r snd $\boldsymbol{\omega}$ are polar co-ordinates with origin at the source, A is a constant, and β and ψ are to be found.

To find β and ψ we first substitute (10.2) into Laplace's equation (6.1) and obtain

$$B\psi = -\beta(\beta+1)\psi \quad \text{in} \quad \Gamma. \quad (10.3)$$

Here B is the angular part of the Laplacian. Next we insert (10.2) into the boundary condition (6.2) on the hull, and expand it for r small, and into the boundary condition (6.4) on $z = 0$. These two conditions can be written together as a single condition on $\partial\Gamma$, the boundary of Γ:

$$\partial\psi/\partial n = 0 \quad \text{on} \quad \partial\Gamma. \quad (10.4)$$

The problem (10.3), (10.4) determines a sequence of eigenvalues β and corresponding normalized eigenfunctions ψ. The smallest eigenvalue satisfies the condition $\beta > 1$ because Γ subtends a solid angle less than 2π. It is this eigenvalue and the associated eigenfunction which determine the leading term in (10.2). The corresponding expansion coefficient of ϕ with respect to $r^{\beta}\psi(\boldsymbol{\omega})$ is the constant A in (10.2).

We now assume that the first limit in (10.1) is given in terms of this constant A by

$$\lim_{\sigma_0\to 0} f(\sigma_0)\, a(\sigma_0) = E(\alpha)\, A. \quad (10.5)$$

We shall call the function $E(\alpha)$ the excitation coefficient of the source. We expect it to depend upon the angle α at which the ray leaves the source and upon the shape of the hull near the source. By using (10.5) in (10.1) we obtain finally

$$a(\sigma) = E(\alpha)\, A \lim_{\sigma_0\to 0} \frac{1}{f(\sigma_0)} \exp\left\{-2\int_{\sigma_0}^{\sigma} (Ds)^2 [\tfrac{1}{2}(Ds)^{-2}\Delta s - D^2 s - i\gamma Ds + \nabla s\,.\,\nabla\eta_0 - \eta_0 s_{zz}]\, d\sigma'\right\}. \quad (10.6)$$

To determine $E(\alpha)$ for a source on a particular hull, we must solve a special problem, which we call a *canonical problem*, of wave production by a special hull. That hull must have the same local geometry near the source as the hull under consideration, and the incident flow must be given by the first term on the right side of (10.2). Thus the cone tangent to the hull at the source must be the same in the canonical problem as in the actual problem, so the canonical hull could be this cone. This way of finding E, by considering a canonical problem, corresponds to using the method of matched asymptotic expansions. The canonical problem yields the first term in the inner expansion.

We have now completed the derivation of the ray theory which was described in §3. As we pointed out there, the problem of finding E is still unsolved in general. We have determined $E(\alpha)$ for a thin ship in §4, but we have not found it for other cases. It can be found by experiment or by numerical solution of canonical problems.

Even with E unknown, the theory describes the structure of the wave pattern in an intuitively clear way. The phase s can be determined readily by solving (3.2) for a ship of any shape. From it the wavefronts and the boundary of the wave pattern can be found. We shall illustrate how to do this in §12 for the special class of streamlined ships which we shall now introduce.

11. Streamlined ships

In the preceding theory, the wave motion was found to satisfy equations obtained by linearizing the problem about $\overline{\phi} = \phi + F^2\phi_1$ and $\overline{\eta} = \eta_0 + F^2\eta_1$. Here ϕ is the double body potential, η_0 is the corresponding surface elevation, while $F^2\phi_1$ and $F^2\eta_1$ are small corrections. We now ask whether there are cases in which it is valid to linearize instead about the state of rest $\overline{\phi} = 0$, $\overline{\eta} = 0$. The reason for asking this question is that the resulting equations will be simpler in such cases, as we shall see. The answer to the question is also quite simple.

Linearization about a state of rest can be expected to be valid if and only if the ship produces a small perturbation of the state of rest. The size of the perturbation which a ship causes is determined by the component of the ship velocity normal to the hull. For ship motion along the x axis, the velocity normal to the hull is proportional to n_x, the x component of the unit normal to the hull. Therefore the perturbation will be small if n_x is small. We shall say that a ship is *streamlined* if n_x is small everywhere on the hull. Thus we conclude that linearization about a state of rest is valid for streamlined ships, and only for them.

From this conclusion we must exclude the neighbourhood of any point on the hull at which n_x is not uniquely defined. At such a point, which may occur at the bow or stern, the flow may have a stagnation point, and there the perturbation is not small. Such exceptional points do not occur if the bow and stern are cusped.

Streamlined ships include the previously studied cases of thin, flat, yacht-shaped and slender ships, as well as many others. However for the general streamlined ship the boundary condition must be satisfied on the actual hull, and not on the centre-plane, centre-line, or other simpler surface as in the previously mentioned special cases. This is necessary because the beam may be large compared to the wavelength $\lambda = 2\pi U^2/g$ even though it must be small compared to the ship length.

The linearized boundary value problem for the perturbation ϕ', η' is given by

(7.3)–(7.7) with $\bar{\phi} = 0$ and $\bar{\eta} = 0$ in the free surface conditions (7.6) and (7.7), and with v on the right side of (7.4). Thus for streamlined ships ϕ' and η' satisfy the following equations:

$$\Delta\phi' = 0, \quad -h \leqslant z \leqslant 0; \tag{11.1}$$

$$\partial\phi'/\partial n = v \quad \text{on} \quad B(t); \tag{11.2}$$

$$\partial\phi'/\partial n = 0 \quad \text{on} \quad z = -h; \tag{11.3}$$

$$\phi'_z = F^2\eta'_t \quad \text{on} \quad z = 0; \tag{11.4}$$

$$\eta' = -\phi'_t \quad \text{on} \quad z = 0. \tag{11.5}$$

Brard (1972) and Noblesse (1978) have considered the problem (11.1)–(11.5) for the steady state in water of infinite depth, but they did not discuss the conditions under which it is appropriate.

Although the discussion given above shows that (11.1)–(11.5) are valid for a streamlined ship, we shall now indicate how these equations can be derived formally. To do so we write the equation of the hull in the original dimensional variables as

$$y'/B = \pm f(x'/L, z'/B)$$

for $0 < x' < L$. Here B is a typical transverse dimension of the hull, such as the beam or draught, L is the length, and f is a dimensionless function. Now we set $B/L = \epsilon$ and $(x, y, z) = (x', y', z')/B$. Then we can rewrite the hull equation as $y = \pm f(\epsilon x, z)$ for $0 < x < L/B = \epsilon^{-1}$. From this equation we find that $n_x = -\epsilon f_1[1 + f_z^2 + \epsilon^2 f_1^2]^{-\frac{1}{2}}$, where f_1 is the derivative of f with respect to its first argument. If the ship is moving along the x axis with velocity $UV(t)$ where $V(t)$ is dimensionless, then the component of this velocity normal to the hull is ϵUv, where

$$\epsilon v = n_x V(t) = -\epsilon V(t) f_1[1 + f_z^2 + \epsilon^2 f_1^2]^{-\frac{1}{2}}. \tag{11.6}$$

From (11.6) we see that ϵv is indeed small, of order ϵ.

We next replace L by B in (5.1) and in the definition of F. Then we assume that the potential Φ and the surface elevation η are of the forms $\Phi = \epsilon\phi' + O(\epsilon^2)$ and $\eta = \epsilon\eta' + O(\epsilon^2)$. We insert these forms into (5.2)–(5.5) and consider the terms of lowest order in ϵ. They yield exactly (11.1)–(11.5) with v in (11.2) denoting the coefficient of ϵ on the right side of (11.6). If $\partial\phi'/\partial n$ in (11.2) is written out in terms of components, the term $\epsilon\phi'_x$ occurs, so it can be omitted. However the main point is that (11.2) holds on the actual hull.

12. Ray theory for streamlined ships

To derive the ray theory for a streamlined ship, we seek the asymptotic forms of ϕ' and η' for F small. We expect the solution to consist of a non-wave-like part which is significant near the ship, and a wave-like part ϕ'_w, η'_w which is dominant far from the ship. We assume that the wave-like part is of the same form as the wave-like part in (7.1) and (7.2), but with $s(\mathbf{x}, F^2\eta, t)$ replaced by $s(\mathbf{x}, 0, t)$ in (7.2). Thus we write

$$\phi'_w = F^3 \exp[iF^{-2}s](a + F^2 a_1) + \dots, \tag{12.1}$$

$$\eta'_w = F \exp[iF^{-2}s(\mathbf{x}, 0, t)](b + F^2 b_1) + \dots\ . \tag{12.2}$$

To determine this solution we shall use (11.1) and (11.3)–(11.5). We shall not use the inhomogeneous equation (11.2) because that involves the non-wave-like part of the solution, which we shall not determine. As a consequence the power of F occurring as a factor in (12.1) is not determined, but will be determined by the appropriate excitation coefficient.

The result of substituting (12.1) and (12.2) into the above equations is the same as setting $\phi = \phi_1 = \eta_0 = 0$ in (7.13), (7.14) and (7.16)–(7.19). Then (8.1)–(8.4) for s and (9.1)–(9.9) for a and b also hold with this same simplification. Furthermore for a constant velocity ship, in co-ordinates fixed in the ship, the ray equations become (3.5)–(3.7) with $\phi = x$. Thus the rays are just the straight lines determined in §4, and the results (4.1)–(4.8) hold for them. However in the present case there are other rays in addition to those from the bow and stern. One ray is produced at each point on the rear portion of the water-line, as we shall now show.

To find these rays we seek a solution for s with $s = 0$ on the water-line. Then $\mathbf{k} = \nabla s$ is normal to the water-line, (4.2) determines the magnitude of $\mathbf{k}$, and (4.1) determines the ray direction. The rays are straight lines, with the angle α between a ray and the x axis given by (4.5). Since these are the rays in a uniform flow, they are the same as those in the Kelvin point ship problem. Then the analysis of those rays, as given for example by Whitham (1974, pp. 410–414), is applicable. It shows that k, α and the angle μ between the directions of $\mathbf{k}$ and the ray are given by

$$k = \sec^2\psi, \quad \tan\alpha = \tan\psi(1+2\tan^2\psi)^{-1}, \quad \tan\mu = -2\tan\psi. \tag{12.3}$$

Here ψ is the acute angle between $\mathbf{k}$ and the negative x axis. The value of s at a point a distance $\sigma-\sigma_0$ along the ray from the water-line is

$$s = (k\cos\mu)(\sigma-\sigma_0). \tag{12.4}$$

In the present case, $\mathbf{k}$ is normal to the water-line so ψ is the acute angle between this normal and the negative x axis, which is just the direction of ship motion. Then $\tan\psi > 0$ and, from the last equation in (12.3), $\tan\mu < 0$ and $\mu > \frac{1}{2}\pi$. Therefore $\cos\mu < 0$ and (12.4) yields $s \leqslant 0$. On the forward part of the water-line, a ray making an angle $\mu > \frac{1}{2}\pi$ with the normal and directed toward the rear will penetrate the hull. Therefore *no rays can be produced on the forward part of the water-line*, except at the bow. However on the rear half of the water-line such rays do not penetrate the hull, so they are produced there. Furthermore $\mu+\psi < \pi$, so these rays are directed toward the x axis, which they cross at some distance behind the stern. Thus the rays from each side of the rear part of the water-line, after crossing the x axis, determine the wave pattern on the *opposite* side of the ship.

The phase on these rays is given by (12.4) and the amplitude by (4.8). Since neighbouring rays intersect at some point, we must measure σ from this point in order that (4.8) hold. Then σ_0 denotes the distance along a ray from this point to the water-line. Upon using these results in (12.2) we obtain

$$\eta'_w = -iF\sec\psi\, a(\sigma_0)\,(\sigma_0/\sigma)^{\frac{1}{2}}\exp\left[iF^{-2}\sec^2\psi\cos\mu(\sigma-\sigma_0)\right]+\dots\,. \tag{12.5}$$

Here we have used (12.3) for k and put $k_1 = k\cos\psi = \sec\psi$. The amplitude factor $a(\sigma_0)$ in (12.5) is so far undetermined. It could turn out to depend upon F, as we indicated above. We shall not pursue this analysis further in this paper.

13. 'Parabolic' equation for waves made by streamlined ships

We shall now describe another method for solving (11.1)–(11.5) to find the waves produced by a streamlined ship. We assume that the ship is moving with constant speed $-U$ along the x axis, and consider co-ordinates with origin fixed in the ship. Then (11.4) and (11.5), which yield $\phi'_z = -F^2\phi'_{tt}$ on $z = 0$ in the original co-ordinates, instead yield

$$\phi'_z = -F^2\phi'_{xx} \quad \text{on} \quad z = 0. \tag{13.1}$$

Thus we seek a solution $\phi'(x,y,z)$ of (11.1)–(11.3) and (13.1) which tends to x at infinity and has no waves ahead of the ship. In these co-ordinates the normal component of the ship velocity is $v = 0$ in (11.3).

To solve these equations we set $k = F^{-2}$ and write

$$\phi'(x,y,z) = x + \exp(ikx + kz)\, u(x,y,z). \tag{13.2}$$

Then (11.1), (11.2) and (13.1) become the following equations for u:

$$2iku_x + 2ku_z + \Delta u = 0, \quad z \leqslant 0, \tag{13.3}$$

$$\frac{\partial u}{\partial n} + ku\,\frac{\partial(ix+z)}{\partial n} = -\frac{\partial x}{\partial n}\exp[-ikx - kz] \quad \text{on} \quad B, \tag{13.4}$$

$$2iku_x + ku_z + u_{xx} = 0, \quad z = 0. \tag{13.5}$$

We now drop u_{xx} from (13.3) and (13.5) because u_{xx} is presumably small compared to ku_x, especially for k large (i.e. F small). Then (13.3)–(13.5) become

$$2iku_x + 2ku_z + u_{yy} + u_{zz} = 0, \quad z \leqslant 0, \tag{13.6}$$

$$\partial u/\partial n + (in_x + n_z)\, ku = -n_x \exp[-ikx - kz] \quad \text{on} \quad B, \tag{13.7}$$

$$2iu_x + u_z = 0, \quad z = 0. \tag{13.8}$$

The result (13.6) is the desired 'parabolic' or Schrödinger-like equation for u, with x playing the role of time. It can be solved by starting at the bow with $u = 0$, and marching toward the stern. In (13.7) n_x and n_z are components of the unit normal to the hull. The term $n_x u_x$ can be omitted from $\partial u/\partial n$ because it is small, as was pointed out above. The reason why u_{xx} is small is that the main x dependence of $\phi' - x$ is presumably contained in the exponential factor.

A further simplification of this problem is possible if u_{zz} is small compared to ku_z. Then we subtract $2k$ times (13.8) from (13.6) evaluated on $z = 0$, and omit u_{zz} to obtain

$$2iku_x - u_{yy} = 0, \quad z = 0. \tag{13.9}$$

We also use (13.8) to eliminate u_z from (13.7) evaluated on $z = 0$, i.e. on the water-line C. Then (13.7) becomes

$$(n_x - 2in_z)\, u_x + n_y u_y + (in_x + n_z)\, ku = -n_x e^{ikx} \quad \text{on} \quad C. \tag{13.10}$$

Now (13.9) is a 'parabolic' equation for $u(x,y,0)$ on the surface $z = 0$ outside the water-line and it can again be solved by marching in x. The boundary condition (13.10) on C can be simplified when n_x and n_z are small.

Different 'parabolic' equation problems can be obtained by replacing x by the double body potential ϕ_0 in (13.2). Then the right sides of (13.7) and (13.8) become zero and $-\phi_{0xx}e^{-ikx}$ respectively. Furthermore, the right sides of (13.9) and (13.10) become $-2k\phi_{0xx}e^{-ikx}$ and zero respectively.

The reason we have omitted the bottom boundary condition (11.3) is that $\phi' - x$ has the exponentially small factor e^{kz}, so that condition is automatically fulfilled when kh is large.

14. Conclusion and relation to other work

Let us now summarize our results and relate them to those of other authors. In §§2–10 we developed a ray theory to describe the waves produced by a slow ship, i.e. one for which F is small. In §11 we introduced the class of streamlined ships and derived the equations governing their waves, while in §12 we obtained the ray theory for them, valid for F small. Section 13 presents another method for obtaining the waves produced by a streamlined ship, which is also particularly useful for F small.

Other recent work on wave making at low Froude number is based on Ogilvie's (1968) study of wavemaking by a submerged two-dimensional body moving at small F. In particular Baba & Takekuma (1975), Baba (1976), Baba & Hara (1978), Newman (1976) and Maruo (1976) used his assumptions, applying them to a ship in three dimensions. In our notation these assumptions are

$$\Phi = \phi + F^{2n}\phi', \quad \nabla\phi = O(1), \quad \nabla\phi' = O(F^{2n-2}), \quad n = 2. \tag{14.1}$$

Comparison of (14.1) with (7.1) shows that ϕ' corresponds to our wave term, but that the two expressions for Φ differ in two respects. First, in (7.1) we have $n = \frac{3}{2}$ while (14.1) has $n = 2$. Secondly we have a term $F^2\phi_1$ in (7.1) which is missing from (14.1).

Concerning the first point, Keller & Ahluwalia (1976) and others have shown that $n = \frac{3}{2}$ is correct for a thin ship according to the Michell theory. Since (14.1) cannot yield that result, we must consider how the value $n = 2$ was arrived at. Newman (1976) notes that if $n < 2$ then ϕ' would satisfy a homogeneous linear equation with linear boundary conditions, and therefore ϕ' would be zero. But the homogeneous linear equation is probably not valid in certain boundary layers near the hull, such as the neighbourhoods of the bow and stern, and other singular points of ϕ. Then ϕ' could be generated in those layers, and it could propagate outward according to the homogeneous equation mentioned above. If this is so, the argument that n cannot be less than two is invalid.

Furthermore the value $n = 2$ and the equation for ϕ' are obtained by requiring terms linear in ϕ' to balance terms involving ϕ. Since ϕ' is a rapidly varying function of position according to (14.1), it is not possible for any linear terms in ϕ' to balance the slowly varying terms involving ϕ. This casts doubt on the value $n = 2$ and on the equation for ϕ'.

This leads us to the second point mentioned above, the absence of the term $F^2\phi_1$ from (14.1). That term is certainly necessary outside the wave region, where ϕ' is negligible, in order to satisfy the boundary conditions on the free surface. By continuity it is likely to be present within the wave region also. In fact it is a slowly varying term and does just balance the terms in ϕ mentioned above. Since the ϕ_1 term affects the variation of wave amplitude along a ray, theories which omit ϕ_1 cannot yield the correct amplitude.

The equation obtained for ϕ' involves advection of the waves by the flow with potential ϕ, as in our theory. However Baba & Takekuma (1975) and Baba (1976) ultimately replace this by advection by the uniform flow with potential x. Thus the problem these authors really consider is an inhomogeneous form of the problem (11.1)–(11.5), which applies to a streamlined ship. The same applies to the work of Maruo (1977) and Noblesse (1978).

The approximate solution for ϕ' obtained by these authors can be obtained by replacing the exact flow by ϕ in an integral representation of the solution. The approximate solution ultimately involves an integral along the water-line. We have evaluated that integral asymptotically for F small by the method of stationary phase. The result for $\phi'(x, y, 0)$ contains terms which correspond exactly to the rays found in § 12, from points on the rear part of the water-line. However the result also contains terms corresponding to rays from the front part of the water-line, which rays pass right through the ship. Thus the approximate solution is incorrect because it contains these unphysical ray contributions.

The problem (11.1)–(11.5) has been considered by Brard (1972) and Noblesse (1978), who call it the Kelvin–Neumann problem. However they did not derive it, nor consider the class of ships for which it is valid.

Despite the shortcomings pointed out above, the numerical results obtained by the authors mentioned seem to be quite good. Evidently they have obtained good approximations, although they may not be the low Froude number asymptotic form of the exact solution.

Inui & Kajitani (1977) have also presented a ray theory, similar to that in Keller (1974), based on the work of Ursell (1960).

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Appendix A. Examples of the occurrence of waves

We shall now present two examples illustrating the way in which waves arise. These examples have some similarity to the ship wave problem, but they are easier to analyse. The first involves the ordinary differential equation

$$u_{xx} + k^2 u = g(x), \quad 0 \leqslant x. \tag{A 1}$$

Here the constant k and the function $g(x)$ are given with $g(x)$ vanishing rapidly at infinity. We seek a solution $u(x)$ satisfying the boundary condition

$$u_x(0) = 0. \tag{A 2}$$

In addition, we require that u satisfy the radiation condition

$$\lim_{x \to \infty} |u_x - iku| = 0. \tag{A 3}$$

We seek the asymptotic form of the solution in the short wave limit, i.e. for $k \gg 1$.

To find this expansion we might try to write u in the form

$$u(x,k) = \frac{1}{k^2}\sum_{j=0}^{\infty}\frac{1}{k^{2j}}u_j(x). \tag{A 4}$$

Substituting (A 4) into (A 1) and equating coefficients of corresponding powers of k^2 yields

$$u_0 = g(x), \quad u_1 = -u_{0xx} = -g_{xx}, \quad u_2 = -u_{1xx} = g_{xxxx}, \quad \ldots \;. \tag{A 5}$$

Thus

$$u = \frac{g(x)}{k^2} - \frac{g_{xx}(x)}{k^4} + \ldots \;. \tag{A 6}$$

This solution fails to satisfy the boundary condition (A 2) if $g_x(0) \neq 0$. In order to correct it, we must add to it a suitable solution of the homogeneous equation (A 1).

The general solution of the homogeneous equation is $a\,e^{-ikx} + b\,e^{ikx}$. Now the radiation condition (A 3) applied to this solution yields

$$\lim_{x\to\infty}\left|-ika\,e^{-ikx} + ikb\,e^{-ikx} - ika\,e^{ikx} - ikb\,e^{ikx}\right| = \left|-2ika\right| = 0. \tag{A 7}$$

Thus $a = 0$, so we add $b\,e^{ikx}$ to the right side of (A 6) to get

$$u = b\,e^{ikx} + \frac{g(x)}{k^2} - \frac{g_{xx}(x)}{k^4} + \ldots \;. \tag{A 8}$$

Now (A 2) yields $0 = ikb + k^{-2}g_x(0) + \ldots$ so $b = -g_x(0)/ik^3 + \ldots$ and (A 8) becomes

$$u(x,k) = \frac{g(x)}{k^2} - \frac{g_x(0)}{ik^3}\,e^{ikx} - \frac{g_{xx}(x)}{k^4} + \ldots \;. \tag{A 9}$$

We note that the wave term is of order k^{-3}, which is intermediate between the orders of the first two terms in (A 6).

In this example the exact solution is easily found to be

$$u(x,k) = \int_0^\infty \frac{1}{2ik}\left(\exp\left[ik|x-x'|\right] + \exp\left[ik|x+x'|\right]\right) g(x')\,dx'. \tag{A 10}$$

Asymptotic evaluation of this integral for $k \gg 1$ yields exactly (A 9) with the equals sign replaced by the sign of asymptotic equality. The wave term comes from the lower limit of integration via integration by parts, while the other terms come from the neighbourhood of $x' = x$.

The second example concerns the partial differential equation

$$\Delta u + k^2 u = g(\mathbf{x}), \quad \mathbf{x} \text{ in } D. \tag{A 11}$$

Here D is a two-dimensional domain which extends to infinity, and which is bounded internally by a curve C. On C we require that the normal derivative of u vanish,

$$\partial_n u = 0, \quad \mathbf{x} \text{ on } C. \tag{A 12}$$

At infinity we require u to satisfy the radiation condition

$$\lim_{r\to\infty} r^{\frac{1}{2}}\left|\partial_r u - iku\right| = 0. \tag{A 13}$$

We proceed as above, writing u in the form (A 4), substituting into (A 11), and equating coefficients. This yields $u_0 = g$, $u_1 = -\Delta g$, etc. Thus (A 4) becomes

$$u = \frac{g(\mathbf{x})}{k^2} - \frac{\Delta g(\mathbf{x})}{k^4} + \dots \quad . \tag{A 14}$$

Again this solution fails to satisfy the boundary condition (A 12) if $\partial_n g \neq 0$ on C. Therefore, we must add to it a suitable solution of the homogeneous equation. For k large, there are such solutions of the asymptotic form $A(\mathbf{x}) \exp(iks(\mathbf{x}))$. Adding this to the right side of (A 14) yields

$$u = A(\mathbf{x})\, e^{iks(\mathbf{x})} + \frac{g(\mathbf{x})}{k^2} - \frac{\Delta g(\mathbf{x})}{k^4} + \dots \quad . \tag{A 15}$$

Now (A 12) becomes

$$ik(\partial_n s)\, A\, e^{iks} + (\partial_n A)\, e^{iks} + k^{-2}\partial_n g + \dots = 0, \quad \mathbf{x} \quad \text{on} \quad C.$$

In order for the first term to cancel the third term we must have

$$s = 0 \quad \text{and} \quad A = -\frac{\partial_n g}{ik^3\, \partial_n s}, \quad \mathbf{x} \quad \text{on} \quad C. \tag{A 16}$$

We can use the values (A 16) of s and A on C to find s and A everywhere. First we substitute $u \sim A\, e^{iks}$ into the homogeneous form of (A 11) and equate to zero the coefficients of the first two powers of k. This yields

$$(\nabla s)^2 = 1, \tag{A 17}$$

$$2\nabla s . \nabla A + A \nabla s = 0. \tag{A 18}$$

The rays associated with (A 17) are straight lines, and since $s = 0$ on C they are normal to C. If σ denotes distance from C and $\mathbf{n}$ is normal to C at the point $\mathbf{x}_0$ on C, we then have

$$s = \sigma \quad \text{at} \quad \mathbf{x} = \mathbf{x}_0 + \sigma \mathbf{n}. \tag{A 19}$$

The corresponding solution of (A 18) is

$$A = A(\mathbf{x}_0) \left(\frac{\rho}{\rho + \sigma}\right)^{\frac{1}{2}} \quad \text{at} \quad \mathbf{x} = \mathbf{x}_0 + \sigma \mathbf{n}. \tag{A 20}$$

Here ρ is the radius of curvature of C at $\mathbf{x}_0$. The initial value $A(\mathbf{x}_0)$ is given by (A 16), and in it $\partial_n s = 1$ according to (A 19).

We finally use (A 19), (A 20) and (A 16) in (A 15) to obtain the desired result

$$u(\mathbf{x}, k) = \frac{g(\mathbf{x})}{k^2} - \frac{\partial_n g(\mathbf{x}_0)}{ik^3} \left[\frac{\rho(\mathbf{x}_0)}{\rho(\mathbf{x}_0) + \sigma}\right]^{\frac{1}{2}} e^{ik\sigma} - \frac{\Delta g(\mathbf{x})}{k^4} + \dots \quad \text{at} \quad \mathbf{x} = \mathbf{x}_0 - \mathbf{n}\sigma. \tag{A 21}$$

Appendix B. Amplitude variation along a ray for a thin ship

In §4 it is shown that, for a thin ship, the exponent in (9.9) is the integral of

$$-\Delta s + 2(D^2 s)(Ds)^2.$$

The phase, given by (4.6), is $s = 2k_1^4 \sigma$ and $Ds = \partial_x s = k_1$. Thus, $D^2 s = s_{xx} = k_{1x}$. Furthermore, by definition $s_y = k_2$ so $s_{yy} = k_{2y}$, and by (4.3)

$$s_{yy} = k_{2y} = (2k_1^3 - k_1)\, k_2^{-1} k_{1y}. \tag{B 1}$$

To find s_{zz} we differentiate (7.13), which is $(\nabla s)^2 = 0$, with respect to z to get

$$s_z s_{zz} + s_x s_{zx} + s_y s_{zy} = 0.$$

According to (8.11), $s_z = -ik_1^2$ on $z = 0$ so $s_{zx} = -2ik_1 k_{1x}$ and $s_{zy} = -2ik_1 k_{1y}$. Thus, we get for s_{zz}

$$s_{zz} = -\frac{1}{s_z}(s_x s_{zx} + s_y s_{zy}) = \frac{1}{ik_1^2}[k_1(-2ik_1 k_{1x}) + k_2(-2ik_1 k_{1y})] = -2k_{1x} - \frac{2k_2}{k_1} k_{1y}. \tag{B 2}$$

We now combine our results to get

$$-\Delta s + 2(D^2 s)(Ds)^2 = -k_{1x} - (2k_1^3 - k_1) k_2^{-1} k_{1y} + 2k_{1x} + 2k_2 k_1^{-1} k_{1y} + 2k_{1x} k_1^2. \tag{B 3}$$

We shall now express k_{1y} in terms of k_{1x} by using the fact that k_1 is constant along a ray. Thus, $\dot{k}_1 = k_{1x}\dot{x} + k_{1y}\dot{y} = 0$, which yields

$$k_{1y} = -\frac{\dot{x}}{\dot{y}} k_{1x} = (2k_1^3 - k_1) k_2^{-1} k_{1x}. \tag{B 4}$$

To find k_{1x} we differentiate (4.4) with respect to x to get

$$1 = (2 - 12k_1^2) k_{1x}\sigma + (2k_1 - 4k_1^3)\sigma_x, \tag{B 5}$$

$$0 = 2(k_1^4 - k_1^2)^{-\frac{1}{2}} \cdot \tfrac{1}{2}(4k_1^3 - 2k_1) k_{1x}\sigma + 2(k_1^4 - k_1^2)^{\frac{1}{2}}\sigma. \tag{B 6}$$

Upon eliminating σ_x and solving for k_{1x} we find

$$k_{1x} = \frac{k_1^2 - 1}{2k_1^2(3 - 2k_1^2)\sigma}. \tag{B 7}$$

We now use (B 7) and (B 4) in (B 3), with k_2 given by (4.3), and the result is (4.7).

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